Completing the Square: answers

Worked solutions and mark scheme · GCSE Maths · Higher tier

M = method mark, A = accuracy mark (needs the method), B = independent mark, ft = follow through from an earlier wrong answer.

Level 1 · Warm-up 10 marks

Monic quadratics with an even x coefficient.

1 Write $x^2 + 6x$ in the form $(x + a)^2 + b$.

Answer: $(x + 3)^2 - 9$

$x^2 + 6x = (x + 3)^2 - 3^2$ *Halve 6 to get 3*

$= (x + 3)^2 - 9$

Marks B1 $(x + 3)^2$ B1 -9

2 Write $x^2 + 10x + 7$ in the form $(x + a)^2 + b$.

Answer: $(x + 5)^2 - 18$

$x^2 + 10x + 7 = (x + 5)^2 - 25 + 7$ *Halve 10 to get 5; subtract 5^2*

$= (x + 5)^2 - 18$

Marks B1 $(x + 5)^2$ B1 -18

3 Write $x^2 - 8x + 3$ in the form $(x + a)^2 + b$.

Answer: $(x - 4)^2 - 13$

$x^2 - 8x + 3 = (x - 4)^2 - 16 + 3$ *Half of -8 is -4*

$= (x - 4)^2 - 13$

Marks B1 $(x - 4)^2$ B1 -13

4 Write $x^2 - 4x - 5$ in the form $(x + a)^2 + b$.

Answer: $(x - 2)^2 - 9$

$x^2 - 4x - 5 = (x - 2)^2 - 4 - 5$ — *Half of -4 is -2; $(-2)^2 = 4$*

$= (x - 2)^2 - 9$

Marks B1 $(x - 2)^2$ B1 -9

5 $x^2 + 12x + 40 \equiv (x + a)^2 + b$. Find the values of a and b.

Answer: $a = 6,\ b = 4$

$x^2 + 12x + 40 = (x + 6)^2 - 36 + 40$ — *Halve 12 to get 6*

$= (x + 6)^2 + 4$

$a = 6, \quad b = 4$

Marks B1 $a = 6$ B1 $b = 4$

Level 2 · Build

13 marks

Odd coefficients, solving equations and turning points.

6 Write $x^2 + 5x + 2$ in the form $(x + a)^2 + b$.

Answer: $\left(x + \frac{5}{2}\right)^2 - \frac{17}{4}$

$x^2 + 5x + 2 = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 2$ — *Half of 5 is $\frac{5}{2}$*

$= \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{8}{4}$ — *Write 2 as quarters*

$= \left(x + \frac{5}{2}\right)^2 - \frac{17}{4}$

Marks B1 $\left(x + \frac{5}{2}\right)^2$ — allow $(x + 2.5)^2$ B1 $-\frac{17}{4}$ — allow -4.25

7 Write $x^2 - 3x - 1$ in the form $(x + a)^2 + b$.

Answer: $\left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$

$x^2 - 3x - 1 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 1$ — *Half of -3 is $-\frac{3}{2}$*

$= \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$

Marks B1 $\left(x - \frac{3}{2}\right)^2$ — allow $(x - 1.5)^2$ B1 $-\frac{13}{4}$ — allow -3.25

8 Solve $x^2 + 6x - 7 = 0$ by completing the square.

Answer: $x = 1$ or $x = -7$

$(x + 3)^2 - 9 - 7 = 0$ — *Complete the square*

$(x + 3)^2 = 16$

$x + 3 = \pm 4$ — *Square root — keep both signs*

$x = 1$ or $x = -7$

Marks M1 $(x + 3)^2 - 16 = 0$ M1 $x + 3 = \pm 4$ A1 $x = 1$ and $x = -7$

9 Solve $x^2 - 4x - 1 = 0$ by completing the square. Give your answers in surd form.

Answer: $x = 2 \pm \sqrt{5}$

$(x - 2)^2 - 4 - 1 = 0$

$(x - 2)^2 = 5$

$x - 2 = \pm\sqrt{5}$

$x = 2 \pm \sqrt{5}$

Marks M1 $(x - 2)^2 - 5 = 0$ M1 $x - 2 = \pm\sqrt{5}$ A1 $x = 2 \pm \sqrt{5}$

10 By completing the square, find the coordinates of the turning point of $y = x^2 - 6x + 11$.

Answer: $(3,\ 2)$

$x^2 - 6x + 11 = (x - 3)^2 - 9 + 11$

$= (x - 3)^2 + 2$

Turning point $= (3,\ 2)$ — *$(x - a)^2 + b$ has turning point (a, b)*

Marks M1 $(x - 3)^2 \ldots$ A1 $(x - 3)^2 + 2$ A1 $(3,\ 2)$ — ft their completed square

Level 3 · Exam-style 19 marks
Multi-part questions in real GCSE style.

11 Write $2x^2 + 12x + 5$ in the form $a(x + b)^2 + c$.

Answer: $2(x + 3)^2 - 13$

$2x^2 + 12x + 5 = 2(x^2 + 6x) + 5$ *Factor 2 out of the x terms*

$= 2\left[(x + 3)^2 - 9\right] + 5$

$= 2(x + 3)^2 - 18 + 5$ *Multiply the -9 by 2*

$= 2(x + 3)^2 - 13$

Marks M1 $2(x^2 + 6x) + 5$ M1 $2[(x + 3)^2 - 9] + 5$ A1 $2(x + 3)^2 - 13$

12 Solve $x^2+8x+3 = 0$ by completing the square. Give your answers correct to 2 decimal places.

Answer: $x = -0.39$ or $x = -7.61$

$(x + 4)^2 - 16 + 3 = 0$

$(x + 4)^2 = 13$

$x = -4 \pm \sqrt{13}$

$x = -0.39$ or $x = -7.61$ $\sqrt{13} = 3.6055\ldots$

Marks M1 $(x + 4)^2 - 13 = 0$ M1 $x = -4 \pm \sqrt{13}$ A1 -0.39 and -7.61

13 (a) Write $x^2 - 10x + 28$ in the form $(x + a)^2 + b$.

Answer: $(x - 5)^2 + 3$

$x^2 - 10x + 28 = (x - 5)^2 - 25 + 28$

$= (x - 5)^2 + 3$

Marks B1 $(x - 5)^2$ B1 $+3$

(b) Hence explain why $x^2 - 10x + 28 > 0$ for all values of x.

Answer: $(x-5)^2 \geq 0$, so $(x-5)^2 + 3 \geq 3 > 0$.

$(x-5)^2 \geq 0$ *A square is never negative*

$(x-5)^2 + 3 \geq 3 > 0$

Marks B1 States a square is never negative, so the expression is at least 3

14 (a) The curve $y = x^2 + 4x - 12$. Write $x^2 + 4x - 12$ in the form $(x+a)^2 + b$.

Answer: $(x+2)^2 - 16$

$x^2 + 4x - 12 = (x+2)^2 - 4 - 12$

$= (x+2)^2 - 16$

Marks B1 $(x+2)^2$ B1 -16

(b) Write down the coordinates of the turning point of the curve.

Answer: $(-2,\ -16)$

$(x+2)^2 - 16 \Rightarrow (-2,\ -16)$ *Change the sign inside the bracket*

Marks B1 $(-2,\ -16)$ — ft their (a)

(c) Use your answer to part (a) to find where the curve crosses the x-axis.

Answer: $x = 2$ and $x = -6$

$(x+2)^2 = 16$

$x + 2 = \pm 4$

$x = 2$ or $x = -6$

Marks M1 $x + 2 = \pm 4$ A1 $x = 2$ and $x = -6$

15 (a) Write $3x^2 - 12x + 7$ in the form $a(x+b)^2 + c$.

Answer: $3(x-2)^2 - 5$

$$3x^2 - 12x + 7 = 3(x^2 - 4x) + 7$$
$$= 3\left[(x-2)^2 - 4\right] + 7$$
$$= 3(x-2)^2 - 12 + 7$$
$$= 3(x-2)^2 - 5$$

Marks M1 $3(x^2 - 4x) + 7$ M1 $3[(x-2)^2 - 4] + 7$ A1 $3(x-2)^2 - 5$

(b) Hence solve $3x^2 - 12x + 7 = 0$. Give your answers in the form $p \pm \frac{\sqrt{q}}{r}$.

Answer: $x = 2 \pm \frac{\sqrt{15}}{3}$

$$3(x-2)^2 = 5$$
$$(x-2)^2 = \frac{5}{3}$$
$$x = 2 \pm \sqrt{\frac{5}{3}} = 2 \pm \frac{\sqrt{15}}{3}$$

Rationalise: $\sqrt{\frac{5}{3}} = \frac{\sqrt{5}\sqrt{3}}{3}$

Marks M1 $(x-2)^2 = \frac{5}{3}$ A1 $2 \pm \frac{\sqrt{15}}{3}$ — allow $2 \pm \sqrt{\frac{5}{3}}$ for M1 only

Level 4 · Challenge — 19 marks

Grade 8–9: unknown coefficients, proof and maxima.

16 (a) Write $5 - 6x - x^2$ in the form $a - (x+b)^2$.

Answer: $14 - (x+3)^2$

$$5 - 6x - x^2 = 5 - (x^2 + 6x)$$ *Take out -1 from the x terms*
$$= 5 - \left[(x+3)^2 - 9\right]$$
$$= 14 - (x+3)^2$$ $-(-9) = +9$

Marks M1 $5 - (x^2 + 6x)$ M1 $5 - [(x+3)^2 - 9]$ A1 $14 - (x+3)^2$

(b) Hence write down the maximum value of $5 - 6x - x^2$ and the value of x at which it occurs.

Answer: Maximum value 14 when $x = -3$

$(x+3)^2 \geq 0 \Rightarrow 14 - (x+3)^2 \leq 14$

Max $= 14$ when $x + 3 = 0,\ x = -3$

Marks B1 Maximum 14 B1 $x = -3$

17 $x^2 + px + 20 \equiv (x+q)^2 - 5$, where p and q are positive constants. Find the value of p and the value of q.

Answer: $p = 10,\ q = 5$

$(x+q)^2 - 5 = x^2 + 2qx + q^2 - 5$ *Expand the right-hand side*

$q^2 - 5 = 20 \Rightarrow q^2 = 25 \Rightarrow q = 5$ *Compare constants; $q > 0$*

$p = 2q = 10$ *Compare x coefficients*

Marks M1 Expands $(x+q)^2 - 5 = x^2 + 2qx + q^2 - 5$ M1 $q^2 - 5 = 20$ A1 $q = 5$ and $p = 10$

18 Prove that $4x^2 - 12x + 10$ is positive for all values of x.

Answer: $4x^2 - 12x + 10 = 4\left(x - \frac{3}{2}\right)^2 + 1 \geq 1 > 0$

$4x^2 - 12x + 10 = 4(x^2 - 3x) + 10$

$= 4\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 10$

$= 4\left(x - \frac{3}{2}\right)^2 + 1$

$4\left(x - \frac{3}{2}\right)^2 \geq 0 \Rightarrow$ expression $\geq 1 > 0$ *State the reason — it earns the final mark*

Marks M1 $4\left(x^2 - 3x\right) + 10$ or $(2x-3)^2 \ldots$ A1 $4\left(x - \frac{3}{2}\right)^2 + 1$ or $(2x-3)^2 + 1$ A1 Conclusion: square ≥ 0, so expression $\geq 1 > 0$

19 $f(x) = 2x^2 - 8x + 11$. Find the coordinates of the turning point of $y = f(x)$, state whether it is a maximum or a minimum, and hence write down the range of values that $f(x)$ can take.

Answer: Minimum at $(2,\ 3)$; $f(x) \geq 3$

$2x^2 - 8x + 11 = 2(x^2 - 4x) + 11$

$= 2\left[(x-2)^2 - 4\right] + 11 = 2(x-2)^2 + 3$

Turning point $(2,\ 3)$ *Positive x^2 coefficient, so it is a minimum*

$f(x) \geq 3$

Marks M1 $2(x^2 - 4x) + 11$ A1 $2(x-2)^2 + 3$ B1 $(2,\ 3)$, minimum — ft their completed square B1 $f(x) \geq 3$

20 Solve $2x^2 + 6x - 3 = 0$ by completing the square. Give your answers in the form $\frac{a \pm \sqrt{b}}{c}$.

Answer: $x = \frac{-3 \pm \sqrt{15}}{2}$

$2\left(x^2 + 3x\right) - 3 = 0$

$2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4}\right] - 3 = 0$

$2\left(x + \frac{3}{2}\right)^2 = \frac{15}{2} \Rightarrow \left(x + \frac{3}{2}\right)^2 = \frac{15}{4}$

$x + \frac{3}{2} = \pm\frac{\sqrt{15}}{2}$

$x = \frac{-3 \pm \sqrt{15}}{2}$

Marks M1 $2\left(x + \frac{3}{2}\right)^2 - \frac{9}{2} - 3 = 0$ M1 $\left(x + \frac{3}{2}\right)^2 = \frac{15}{4}$ M1 $x + \frac{3}{2} = \pm\frac{\sqrt{15}}{2}$ A1 $\frac{-3 \pm \sqrt{15}}{2}$